Example 10: Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

1. Calculate $\text{rank}(A)$.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{R_2 := R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$

$\text{rank}(A) = 2$

2. Calculate $\text{rank}(A^T)$. What do you observe?

$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \xrightarrow[R_2 := R_2 - 2R_1]{R_3 := R_3 - 3R_1} \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 0 & -6 \end{bmatrix} \xrightarrow{R_3 := R_3 - 2R_2} \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$

$\text{rank}(A^T) = 2$

Theorem 5: Let A be a $m \times n$ matrix. Then

$$\text{rank}(A) = \text{rank}(A^T)$$

Example 11: Fill in the Blanks: Let A be a 12×15 matrix. If $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$, then

$\text{rank}(A) =$ 12 $\qquad$ $\text{nullity}(A) =$ $15 - \text{rank}(A) = 3$

$\text{rank}(A^T) =$ 12 $\qquad$ $\text{nullity}(A^T) =$ $12 - \text{rank}(A) = \emptyset$

$\text{rank}(A^T A) =$ 12 $\qquad$ $\text{nullity}(A^T A) =$ $15 - \text{rank}(A) = 3$

$A =$ 12 × 15 $\qquad$ $A^T =$ 15 × 12 $\qquad$ $A^T A:$ 15 × 15

Example 12: Fill in the Blanks: Let A be a 17×11 matrix. If $A^T A$ is invertible, then

$\text{rank}(A) =$ 11 $\qquad$ $\text{nullity}(A) =$ $11 - \text{rank}(A) = \emptyset$

$\text{rank}(A^T) =$ 11 $\qquad$ $\text{nullity}(A^T) =$ $17 - \text{rank}(A) = 6$

$\text{rank}(A^T A) =$ 11 $\qquad$ $\text{nullity}(A^T A) =$ $11 - \text{rank}(A^T A) = \emptyset$

$A:$ 17 × 11 $\qquad$ $A^T:$ 11 × 17 $\qquad$ $A^T A:$ 11 × 11